Annals of Mathematical Logic 22 (1982) 177–196
North-Holland Publishing Company

SOME APPLICATIONS OF JENSEN'S CODING THEOREM

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Received 8 April 1980

0. Introduction

In this paper we essentially prove*

Theorem 1. *Let M be a transitive model of* ZFC+GCH. *There is an M-definable class P of conditions such that if N is a P generic extension of M, then there is a real r in N such that*:

(1) *N satisfies*: $V=L(r)+\mathrm{ZF}$,

(2) *if α is an ordinal in M, then $L_\alpha(r)$ is not a model of* ZF.

As a corollary of this theorem we shall have the following results.

Theorem 2. *Let α be a countable ordinal such that L_α is a model of* ZF; *there is a real r such that α is the first ordinal β such that $L_\beta(r)$ is a model of* ZF.

And more generally

Theorem 3. *Let $(\alpha_i)_{i<\lambda}$ be a countable sequence of countable ordinals such that for $i<\lambda$*:

(1) $L_{\alpha_i}, (\alpha_j)_{j<i} \models \mathrm{ZF}$,

(2) $\mathrm{Sup}(\alpha_j \mid j<i)<\alpha_i$.

There is a real r such that for $\beta \leqslant \mathrm{Sup}(\alpha_i \mid i<\lambda)$

$$L_\beta(r) \models \mathrm{ZF} \quad \text{iff } \exists\, i<\lambda\ \beta=\alpha_i.$$

Using the methods developed in [2] we can also prove:

Theorem 4. *Let M be a transitive model of* ZF+$V=L$. *There is an M-definable class P of conditions such that if N is a P-generic extension of M, then there is a real*

* After we have written this paper, R. Jensen tells us that A. Beller has proved—independently and before—the Theorem 1 by similar methods in his unpublished thesis.

r in N such that:

(1) *N satisfies*: ZF + $V = L(r) + r$ *is* Π^1_2 *singleton.*

(2) *If* α *is an ordinal in M, then* $L_\alpha(r)$ *is not a model of* ZF.

As a corollary of this theorem we have:

Corollary. *Let M be a countable transitive model of* ZF + $V = L$. *There is a generic extension N of M such that every set in N is N-definable.*

Using the methods developed in [12] and Theorem 1 we also have

Corollary. *Let M be a countable transitive model of* ZFC. *There is a generic extension N of M such that every set in N is N-definable.*

Remarks. (1) Theorems 2 and 3 are the analog for ZF of the theorems of Sacks and Friedman–Jensen for KP (the theory of admissible sets) (for details see [5] and [6]).

(2) Clearly condition (1) in Theorem 3 is necessary. We do not know if the theorem can be proved under a weaker assumption than condition (2). However, if (2) is not true we have to add another condition for the following reason: Let j be such that $\alpha_j = \mathrm{Sup}(\alpha_i \mid i < j)$. Suppose r satisfies the conclusion of the theorem. Let τ be such that: $L_{\alpha_j}(r) \models \tau = \aleph_1$, and i such that $\alpha_i > \tau$. By the Lowenheim–Skolem theorem there is k such that $\alpha_k < \tau$ and there is an isomorphism from $L_{\alpha_k}(r)$ in $L_{\alpha_i}(r)$ whose restriction to L_{α_k} is an isomorphism from L_{α_k} in L_{α_i}. So the α_i cannot be arbitrary.

(3) We also note that, in Theorem 4, the real r remains Π^1_2 in every extension N' of N that preserves $\aleph_1$. The fact that M has to satisfy $V = L$ also is too restrictive. We give, at the end of this paper, some other cases where the conclusion remains true with a weaker hypothesis.

(4) We note that, in Theorem 1, if there is no inaccessible cardinal in M, then cardinals and cofinalities are preserved in the extension. The fact that M has to satisfy GCH is not necessary since we can make it true by a first generic extension.

(5) In the proof of Theorem 1 we shall state some results that are by themselves of some interest (see Theorems 6, 7 and 8).

Finally we state a proposition that is an immediate corollary of [2, Theorem IV] and we feel also of some interest.

Proposition. *Let M be a model of* ZF + $V = L$. *For every cardinal K in M there is a complete and rigid boolean algebra B in M which is countably generated, whose cardinal is greater than K and that satisfies*:

$$[\forall \alpha(\alpha \text{ cardinal in } L \to \alpha \text{ cardinal}) \;\&\; \mathrm{cf}^L(\alpha) = \mathrm{cf}\ \alpha] = \mathbb{1}_B.$$

In this paper we heavily use the following 'fantastic' theorem (and its proof!)

Theorem 5 (R. Jensen). *Let M be a transitive model of* ZFC+GCH. *There is an M-definable class P of conditions such that if N is a P-generic extension of M, then*

(1) *N satisfies*: $\mathrm{ZF}+\exists\, a\subset\omega\ V=L(a)$;

(2) *cardinals and cofinalities are preserved.*

Our proof is divided into two steps. In the first one we ensure that the generic extension N satisfies: $V=L(a)+$there is no *cardinals* α such that $L_\alpha(a)$ satisfies ZF. It only uses results that are stated in [9]. We first use the Theorem 5 to have a model of $V=L(a)$.

Noting that if α is a limit cardinal, then: $L_\alpha\models\mathrm{ZF}$ iff $L_{\alpha+1}\models\alpha$ is regular we first add a class A_0 of cardinals to collapse all the possible inacessible cardinals, then we code by a real, then we add another class A_1 of ordinals such that if α is a limit cardinal, then $L_{\alpha+1}(A_1\cap\alpha)\models\alpha$ is sinular. Then we code another time by a real.

In the second step we ensure the result for all the ordinals. Since this step will preserve cardinals and cofinalities, it is necessary to have done the first step before, and we shall see where this is used in the proof (see Lemma 14, note 1). The proof of this second step will use the method developed in [9]. But we are dealing with a very simple case: we start from L (in fact $L(a)$ with $a\subset\omega$, but a causes no problem) and in addition if α is a limit cardinal $L_{\alpha+1}\models\alpha$ is singular. Since a big part of the proof of Theorem 5 is devoted to cases that do not occur here or are trivial, we shall write some technical details in the proof of our second step (although they are exactly as in [9]) and hope that this will shed some light on the proof of Theorem 5 and also on its motivations.

For a motivation of this second step, the reader may see at the beginning of the second step.

1. The first step

Theorem 6. *Let M be a transitive model of* $\mathrm{ZF}+V=L$. *There is an M-definable class P of conditions such that if N is a P-generic extension of M, then N satisfies*:

(1) $\mathrm{ZF}+\exists\, a\subset\omega\quad V=L(a)$.

(2) *There are no inaccessible cardinals.*

(*Note*: This relativizes to $L(b)$ with $b\subset\omega$.)

Proof. If there are no inaccessible cardinals, there is nothing to do! If M satisfies: $\exists\alpha\,\forall\beta>\alpha\,\beta$ is not an inaccessible cardinal, this is easy: we collapse on $\aleph_1$ a successor cardinal α such that for all $\beta>\alpha$ β is not inaccessible. So let us assume that in M the class of inacessible cardinals is a proper class. Let P_0 be the following class of conditions:

$p \in P_0$ iff p is a function from an ordinal $|p|$ into 2 such that, if $\beta \leq |p|$ and β is inaccessible, then $\{\gamma < \beta : p(\gamma) = 1\}$ is a bounded subset of β.

$p \leq q$ iff $p, q \in P_0$ and $p \supset q$.

Let M_0 be a P_0 generic extension of M.

Claim. (1) *M_0 and M have the same sets.*

(2) *There is a closed unbounded class A of ordinals that contains no inaccessible cardinals (i.e.* On *is not Mahlo).*

Proof. (1) It comes easily from the fact that for every condition p and cardinal α there is a $q \leq p$ such that: $P_0(q) = \{r \in P_0 \mid r \leq q\}$ is (α, ∞) distributive.

(2) The class A is the class of the limit points of $B = \{\gamma \mid \exists\, p \in G_0 \quad p(\gamma) = 1\}$.

Let $(i_\alpha)_{\alpha \in \mathrm{On}}$ be the monotone enumeration of the inaccessible cardinals in M_0. We define (in M_0) the family $(c_\alpha)_{\alpha \in \mathrm{On}}$ by:

$$c_0 = \omega,$$

$$c_1 = \omega_1,$$

$$c_{\alpha+2} = (\text{the first member of } A \text{ greater than } i_\alpha, c_{\alpha+1})^{++}$$

if α is a limit ordinal

$$c_\alpha = \mathrm{Sup}(c_\beta \mid \beta < \alpha); \qquad c_{\alpha+1} = c_\alpha^+.$$

It is clear that no c_α is an inaccessible cardinal. □

It is then enough to show:

Theorem 7. *Let M be a transitive model of* ZF + $V = L$ *and $(c_\alpha)_{\alpha \in \mathrm{On}}$ be an increasing class of cardinals such that: for $\alpha \in \mathrm{On}$*

(1) *$c_{\alpha+1}$ is regular and if α is limit, then $c_\alpha = \mathrm{Sup}_{\beta<\alpha} c_\beta$; $c_0 = \omega$,*

(2) *if $c_{\alpha+2} = \gamma^+$, then $\mathrm{cf}(\gamma) \geq c_{\alpha+1}$,*

(3) *if α is limit, then $c_{\alpha+1} = c_\alpha^+$.*

There is a class P of conditions such that if N is a P generic extension of M, then N satisfies:

(1) ZF + $\exists\, a \subset \omega \quad V = L(a)$.

(2) $\forall \alpha$ *(α is an infinite cardinal iff $\exists \beta\, \alpha = c_\beta$).*

Note: The conditions imposed on the family (c_α) are necessary. The first one is trivial. For the second one: if a is a real $L(a)$ satisfies GCH. So if we had $\aleph_{\alpha+2}^{L(a)} = (\gamma^+)^L$ together with $\mathrm{cf}^L(\gamma) < \aleph_{\alpha+1}^{L(a)}$ we derive a contradiction by:

$$\aleph_{\alpha+1}^{L(a)} = |\aleph_{\alpha+1}^{\mathrm{cf}^L\gamma}|^{L(a)} = |\,|\gamma|^{\mathrm{cf}^L\gamma}|^{L(a)} \geq (\gamma^+)^L = \aleph_{\alpha+2}^{L(a)}.$$

The third condition is not strictly necessary: we write it like that for simplicity. It comes from Jensen's covering theorem that implies that if K is a singular cardinal and $0^{\#}$ does not exist then $(K^+)^L = K^+$.

Proof. We only give a sketch of the proof: there are two parts. In the first one we collapse cardinals by standard methods (details may be found in [8]). In the second one we code this by a real, using Theorem 5.

If λ and β are cardinals we denote:

(1) $L(\lambda, \beta) = \{p : \gamma < \lambda \to \beta\}$ with the order: $p \leqslant q \leftrightarrow p \supset q$. It is easily shown that if λ is regular and $\operatorname{cf} \beta \geqslant \lambda$ then $|L(\lambda, \beta)| = \beta$; $L(\lambda, \beta)$ is $(<\lambda, \infty)$ distributive and if N is a $L(\lambda, \beta)$ generic extension of M, then

(i) N satisfies: GCH $+ \lambda^+ = (\beta^+)^L$,

(ii) cardinals less than λ^+ or greater than β^+ are preserved.

(2) $Q(\lambda, \beta) =$ the λ-sum of the family $L(\lambda, \alpha)$ for $\alpha \in I$ where $I = \{\alpha < \beta \mid \alpha$ cardinal & $\operatorname{cf} \alpha \geqslant \lambda\}$; and the λ-sum of a family of P_α is the subset of the cartesian product given by the families $(p_\alpha)_{\alpha \in I}$ such that $p_\alpha = \mathbb{1}_\alpha$ for all α but a subset of cardinal less than λ.

It is easily shown that if λ is regular and β is inaccessible, then $Q(\lambda, \beta)$ is $(<\lambda, \infty)$ distributive, satisfies the $<\beta$-chain condition and is of cardinal β. Moreover if N is a $Q(\lambda, \beta)$ generic extension of M, then

(i) N satisfies: GCH $+ \beta = \lambda^+$,

(ii) cardinals less than λ^+ or greater than β^+ are preserved.

(3) Finally we denote, for an ordinal α

$$\begin{aligned} C_1 &= L(\omega, \beta) \quad \text{if } c_1 = \beta^+ \\ &= Q(\omega, \beta) \quad \text{if } c_1 = \beta \text{ is inaccessible,} \\ C_\alpha &= L(c_{\alpha+1}, \beta) \quad \text{if } c_{\alpha+2} = \beta^+ \\ &= Q(c_{\alpha+1}, \beta) \quad \text{if } c_{\alpha+2} = \beta \text{ is inaccessible.} \end{aligned}$$

$C = \prod_{\alpha \in \mathrm{On}} C_\alpha$ i.e. $p \in C$ iff $\exists\, \alpha \in \mathrm{On}\ p = (p_\beta)_{\beta < \alpha}$ and $\forall\, \beta < \alpha\ (p_\beta \in C_\beta$ & if $\beta = c_\beta$ is an inaccessible cardinal, then $\exists\, \gamma < \beta\, \forall\, \mu \in [\gamma, \beta[\ p_\mu = \mathbb{1}_\mu)$; $p \leqslant q$ iff $p, q \in C$ & $\operatorname{dom} p \supset \operatorname{dom} q$ & $\forall\, \beta \in \operatorname{dom} q\ p_\beta \leqslant q_\beta$.

It is easily seen that the conditions on $(c_\alpha)_{\alpha \in \mathrm{On}}$ are just what is needed for C to be an Easton forcing and so if N is a C generic extension of M, then N satisfies:

(i) ZFC+GCH,

(ii) $\forall \beta(\beta$ is a cardinal iff $\exists \alpha\, \beta = c_\alpha)$.

An application of Theorem 5 achieves the proof. $\square$

Now Theorem 6 ensures that we may suppose there is no inaccessible cardinal in M. The following result shows that we may suppose there is no *cardinal* α in M such that $L_\alpha \models \mathrm{ZF}$.

Theorem 8. *Let M be a transitive model of* ZF $+ V = L +$ *there is no inaccessible cardinal. There is an M-definable class P of conditions such that if N is a P-generic* *extension of M, then*

(1) *N satisfies*: $\mathrm{ZF}+V=L(a)$ *for* $a\subset\omega$.

(2) *There is no cardinal* α *such that* $L_\alpha(a)\models\mathrm{ZF}$.

(3) *Cardinals and cofinalities are preserved.*

(*Note*: This relativizes to $L(b)$ with $b\subset\omega$.)

Proof. As in the proof of Theorem 6 we first add a class of ordinals A such that if α is a limit cardinal, then $L_{\alpha+1}(A\cap\alpha)\models\alpha$ is singular, and then use Theorem 5 to code A by a real a. Since the proof of Theorem 5 shows that if α is a limit cardinal, then $A\cap\alpha\in L_{\alpha+1}(a)$ we are done. We now describe the class that does the job: $p\in P$ iff p is a function from an ordinal $|p|$ into 2 such that for $\xi\leqslant|p|$, if ξ is a limit cardinal, then $L_{\xi+1}(p\restriction\xi)\models\xi$ is singular. $p\leqslant q$ iff $p,q\in P$ & $p\supset q$. $\square$

Claim 1: $\forall p\in P\,\forall\xi\in\mathrm{On}\,(\xi\geqslant|p|\rightarrow\exists q\leqslant p\ |q|=\xi)$.

The only non trivial case is when ξ is a limit cardinal. Let $\lambda=\mathrm{cf}\,\xi<\xi$. We may suppose $|p|\geqslant\lambda$. Let $(\xi_i)_{i<\lambda}$ be a monotone normal sequence of cardinals less than ξ converging to ξ. Define $(p_i)_{i\leqslant\lambda}$ such that $|p_i|=\xi_i$ by $p_0=p$. Let $q_i\leqslant p_i$ be such that $|q_i|=\xi_{i+1}$; define p_{i+1} by: $p_{i+1}\restriction(\xi_i,\xi_{i+1}(\ =r$ where $r(\langle\alpha,0\rangle)=1$ iff $\alpha=\langle i,\xi_i\rangle$; $r(\langle\alpha,1\rangle)=q_i(\alpha)$; $r(\langle\alpha,\beta\rangle)=0$ for $\beta\geqslant2$ (where $\langle\ ,\ \rangle$ is the Gödel bijection from $\mathrm{On}\times\mathrm{On}$ onto On).

If i is a limit ordinal $p_i=\bigcup_{j<i}p_j$.

It is clear that $p_i\in P$ for $i\leqslant\lambda$ and so that $|p_\lambda|=\xi_\lambda=\lambda$ and $p_\lambda\leqslant p$.

Claim 2. *Let* α *be a cardinal in M, then P is* (α,∞) *distributive, that is: Let* $(\Delta_i)_{i<\alpha}$ *be a family of strongly dense subclasses of P (i.e.* $\{(i,p)\mid p\in\Delta_i\ \&\ i<\alpha\}$ *is a class in M and* $\forall i\,\Delta_i$ *is strongly dense), then* $\bigcap_{i<\alpha}\Delta_i$ *is strongly dense.*

From these two claims it easily follows that P does the job.

Proof. Let $p\in P$. We define $(p_i,\beta_i)_{i\leqslant\alpha}$ by: $p_0=p$; β_0 is the first ordinal β such that $\beta>|p_0|$ & $\beta>\alpha$ & L_β contains all the parameters in the definition of $(\Delta_i)_{i<\alpha}$; β_{i+1} is the first ordinal $\beta>|p_i|$ such that L_β reflects the formula '$p\in\Delta_i$' and all its subformulas (where reflects means: $\forall q\in L_\beta\ L_\beta\models q\in\Delta_i$ iff $q\in\Delta_i$); p_{i+1} is the L-least condition q such that: $q\in\Delta_i$ & $|q|>\beta_{i+1}$ & $q\leqslant p_i$ for limit i: $\beta_i=\mathrm{Sup}_{j<i}\beta_j$ $p_i=\bigcup_{j<i}p_j$ if $\bigcup p_j$ is a condition; undefined if not.

It is then enough to prove that for $i\leqslant\alpha$ p_i is defined. It is clear for i a successor. If i is limit: $\beta_i=\mathrm{Sup}\,\beta_j=|\bigcup p_j|$.

It is well known that the supremum of a sequence of ordinals which reflects a formula and all its subformulas does the same thing. So the sequence $(p_j,\beta_j)_{j<i}$ is definable in L_{β_i} in the same way it was defined in M. So $L_{\beta_i+1}\models\beta_i=|p_i|$ is singular. Since it is clear that for $\xi<|p_i|$ $L_{\xi+1}(p\restriction\xi)\models\xi$ is singular, the claim is proved. $\square$

Now we may suppose that M is a model of $ZF+V=L$ and that for no cardinal α in M $L_\alpha \models ZF$. (In fact we must deal with $V=L(a)$ for some real a but since a causes no problem we omit it.)

Moreover we assume that $\{\beta \mid L_\beta \models ZF\}$ is a proper class in M (otherwise the result is trivial) and so (by the Lowenheim–Skolem theorem) that for every cardinal α in M $\{\beta<\alpha \mid L_\beta \models ZF\}$ is unbounded in α. □

2. The second step

Let us first give some motivation. Let R be the following set: $p \in R \leftrightarrow p:|p|<\aleph_1 \to 2$ such that.

$$\forall \xi \leq |p| \, \forall \beta \, (L_\beta(p \restriction \xi) \models ZF \to L_\beta \models \overline{\overline{\xi}} = \omega).$$

Suppose $D \subset \aleph_1$ be such that: $\forall \xi<\aleph_1$ $D \cap \xi \in R$; code, by use of the almost disjoint sets, D by a real a; then for $\alpha<\aleph_1$ $L_\alpha(a) \not\models ZF$ (if not assume $\alpha<\aleph_1$ and $L_\alpha(a) \models ZF$; let ξ be $\aleph_1$ in L_α; it is clear that $D \cap \xi \in L_\alpha(a)$ but this is a contradiction since then $L_\alpha(D \cap \xi) \models ZF$ and $L_\alpha \models \xi = \aleph_1$).

Moreover it can be shown that if $\mu>\aleph_1$ and $L_\mu \models \forall \alpha>\aleph_1$ $L_\alpha \neq ZF$ and D is R generic over L_μ, then $\aleph_1$ is preserved in $L_\mu(D)$.

Now the method is: Iterate this forcing just as the proof of Theorem 5 is an iteration of the forcing with almost disjoint sets.

Remark. A sceptical reader may think: since in the forcing R we may always use the same finite conjunction of ZF axioms to have $L_\beta(p) \not\models ZF$ if $L_\beta \models |p| \geq \aleph_1$ and $\beta<\aleph_1$ (for example the axioms that imply that every well-ordered set is isomorphic to an ordinal), the same proof will give a model where a finite (and precise) subset of the ZF axioms is false for every $L_\beta(a)$ (since we use in our proof only a finite number of the ZF axioms) contradicting, then, the reflection principle! We will show (Lemma 14, note 1) that we cannot assume (happily!) that it is always the same axioms we use.

We now begin the proof: Our notations will be essentially that of [9]. Set Card = the class of cardinals α such that $\alpha=0$ or α is infinite (we accept the convention $\omega=0^+$. We note $S_0=\{p \mid |p|<\omega \to 2\}$. For $\alpha \in$ Card, $\alpha \geq \omega$ let S_α be the set of functions $p:|p| \to 2$ such that $|p| \in (\alpha, \alpha^+($ and for $\beta \in On$, for $\xi \leq |p|$ if $L_\beta(p \restriction \xi) \models ZF$, then $L_\beta(p \cap Z \restriction \xi) \models \overline{\overline{\xi}} = \alpha$ (where $Z=\bigcup_{\nu \neq 0} Z_\nu$ with $Z_\nu=\{\langle \xi, \nu \rangle \mid \xi \in On\}$ and $\langle \ , \ \rangle$ is the Gödel pairing function).

(*Note*: in [9] Z_0 was used to code A (where $M \models V=L(A)$, $A \subset On$), here we use it to deny ZF).

Definition. Let $\alpha \geq \omega$, by induction on $|p|$ ($p \in S_\alpha$) define μ_p as follows: μ_p is the first ordinal μ such that: $\mu>\alpha$ $\mu>\mathrm{Sup}(\mu_\xi \mid \xi \in [\alpha, |p|[)$ and $L_\mu(p) \models ZF^- + \forall x \ \bar{\bar{x}} \leq \alpha$.

Definition. Let $\alpha > \omega$, $p \in S_\alpha$ we set: $A_p = (L_{\mu_p}(p); p)$ Recall that if α is a limit cardinal, then $A_p \models \alpha$ is singular (if $p = \phi$ set A_α instead of $A_\emptyset$).

Lemma 1. *Let α be a successor cardinal (including ω by our convention); there is a sequence $(b_p \mid p \in S_\alpha)$ such that:*

(1) $b_p \subset \alpha$.

(2) $b_p \in L_{\mu_{p\restriction 0}}$ *and is uniformly definable from* p.

(3) *Let $\delta < \alpha$ and $f : \delta \to [\xi, |p|[$ be injective; then $(b_{p \restriction f(i)} \mid i < \delta)$ is Cohen generic over A_p.*

Proof. This is [9, §3, Lemma 3].

Definition. For $b \subset \alpha$ set $S(b)$ = the set of $(\eta, 1)$ such that η is a code for $b \restriction \xi$ for some $\xi < \alpha$.

Definition. Let $s \in S_{\alpha^+}$; we set R^s = the set of pairs $(\dot r, r)$ such that:

(1) $r \in S_\alpha$ (we denote $\bar r = \{\xi \mid r(\xi) = 1\}$),

(2) $\bar{\bar r} \leqslant \alpha$ and $\dot r \subset (P_{<\alpha^+}(\alpha^+) \cup \{b_{s \restriction \xi} \mid s(\xi) = 1\}) \times (\alpha, \alpha^+($,

(3) if $(b, \eta) \in \dot r$, then $(S(b) - \eta) \cap \bar r = \emptyset$.

$(\dot r, r) \leqslant (\dot p, p)$ iff $\dot r \supset \dot p$ & $r \supset p$.

The following is the key of the proof.

Lemma 2. *Let $s \in S_{\alpha^+}$ then*

(1) *R^s is (α, ∞) distributive in A_s,*

(2) *R^s satisfies the $\leqslant \alpha^+$ chain condition in A_s,*

(3) *if G is R^s generic over A_s there is a $D \subset (\alpha, \alpha^+($ such that: $A_s(G) = L_{\mu_s}(D)$ and $s \in L_{\mu_s}(D \cap Z_0)$,*

(4) *moreover we have for β, $\xi \in [\alpha, \mu_s[$ if $L_\beta(D \cap Z \cap \xi) \models \mathrm{ZF} + \xi = \alpha^+$ then $L_\beta(D \cap \xi) \not\models \mathrm{ZF}$.*

Proof. Forcing with R^s is equivalent to the two step forcing:

(i) Add a set $D_0 \subset [\alpha, \alpha^+[\cap Z_0$ to code s: this is trivially (α, ∞) distributive and satisfies the chain condition.

Claim. *$A_s(D_0) = L_{\mu_s}(D_0)$ satisfies: $\forall \beta > \alpha^+$ $L_\beta(D_0) \not\models \mathrm{ZF}$.*

Let β be such that $\alpha^+ < \beta < \mu_s$ and $L_\beta(D_0) \models \mathrm{ZF}$; begin to recover (in $L_\beta(D_0)$) s: a $\xi \leqslant |s|$ such that $L_\beta(s \restriction \xi) \models \xi = \alpha^{++}$ cannot exist for if ξ is the first one, then $s \restriction \xi \in L_\beta(D_0)$ and so $L_\beta(s \restriction \xi) \models \mathrm{ZF}$ and $L_\beta(s \cap Z \cap \xi) \models \xi \geqslant \alpha^{++}$ a contradiction; so $s \in L_\beta(D_0)$ and $L_\beta(D_0) \models |s| < \alpha^{++}$. But then we can define μ_s in $L_\beta(D_0)$ and so $\mu_s < \beta$.

(ii) Forcing with the following set of conditions:

$R = \{p \mid p : |p| \cap Z_0 \to 2$ such that $|p| \in [\alpha, \alpha^+[$ and for $\xi \leqslant |p|$ if $L_\beta(D_0 \cap \xi, p \restriction \xi) \vDash \mathrm{ZF}$, then $L_\beta(D_0 \cap \xi) \vDash \underline{\underline{\xi}} = \alpha\}$; $p \leqslant q$ iff $p, q \in R$ and $p \supset q$.

Claim. *R is (α, ∞) distributive in $A_s(D_0)$.*

Let $(\Delta_i)_{i<\alpha}$ be a sequence, in $A_s(D_0)$, of strongly dense subsets of R, and $p \in R$; set $b = A_s(D_0)$; define $(X_i)_{i \leqslant \alpha}$ as follows:

$X_0 =$ the smallest $X \prec b$ such that $\alpha \cup \{p, (\Delta_i \mid i < \alpha)\} \subset X$,

$X_{i+1} =$ the smallest $X \prec b$ such that $X_i \cup \{X_i\} \subset X$.

$X_\lambda = \bigcup_{i<\lambda} X_i$ for limit λ.

Set $\sigma_i : b_i \xrightarrow{\sim} X_i$, b_i transitive; then $b_i = L_{\delta_i}(D_0 \cap \alpha_i)$ where $\alpha_i = X_i \cap \alpha^+ = \sigma_i^{-1}(\alpha^+)$.
Define a sequence p_i in R by:

$p_0 = p$,

$p_{i+1} =$ the $L_{\alpha^+}(D_0)$—least $q \leqslant p_i$ such that $|q| \geqslant \alpha_i$ and $q \in \Delta_i$.

(It is clear that: $\forall p \in R \quad \forall \xi < \alpha^+ \quad \exists q \leqslant p \; |q| \geqslant \xi$.)

For limit λ $p_\lambda = \bigcup_{i<\lambda} p_i$ if $\bigcup_i p_i \in R$; undefined if not. It is enough to show that for $i \leqslant \alpha$ p_i is defined; the only non trivial case is when λ is a limit ordinal.

We only have to prove that if $\xi = \alpha_\lambda = |\bigcup_{i<\lambda} p_i|$ and $L_\beta(D_0 \cap \xi, q) \vDash \mathrm{ZF}$, then $L_\beta(D_0 \cap \xi) \vDash \underline{\underline{\xi}} = \alpha$ (we note $q = \bigcup_i p_i$); (if $\xi < |q|$ this is trivial) we know that b satisfies: $\forall \beta > \alpha^+ \; L_\beta(D_0) \nvDash \mathrm{ZF}$; so $L_{\delta_\lambda}(D_0 \cap \xi) \vDash \forall \beta > \xi \; L_\beta(D_0 \cap \xi) \nvDash \mathrm{ZF}$; so a β such that $L_\beta(D_0 \cap \xi, q) \vDash \mathrm{ZF}$ is greater than δ_λ; but then $b_\lambda \in L_\beta(D_0 \cap \xi)$ and since $(\alpha_i, p_i)_{i<\lambda}$ is definable from b, p_0 and $\sigma_i^{-1}((\Delta_i)_{i<\alpha})$, $L_\beta(D_0 \cap \xi) \vDash \underline{\underline{\xi}} = \alpha$ since $\xi = \alpha_\lambda = \mathrm{Sup}_{i<\lambda} \alpha_i$ and $L_\beta(D_0 \cap \xi) \vDash \bar{\bar{\alpha}}_i = \alpha$. This proves part 1 of the lemma.

(2) and (3) are proved by standard ways.

(4) Let $\beta, \xi \in (\alpha, \mu_s($, be such that $L_\beta(D \cap Z \cap \xi) \vDash \mathrm{ZF} + \xi = \alpha^+$. We have to show that $L_\beta(D \cap \xi) \nVdash \mathrm{ZF}$. If $\beta > \alpha^+$ it has already been proved. If $\beta = \alpha$ or α^+ we use the fact that no cardinal in M satisfy ZF. If $\beta \in)\alpha, \alpha^+($ it is immediate from the definition of S_α since $D \cap \xi \in S_\alpha$. □

Definition. Let $\alpha \in \mathrm{Card}$, set $S_\alpha^+ =$ the set of $D \subset (\alpha, \alpha^+($ (in or out of M) such that

(1) $\delta \in (\alpha, \alpha^+(\to D \cap \delta \in S_\alpha$,

(2) $A_D = L_{\alpha^+}(D) \vDash \mathrm{ZF}^-$.

Definition. Let $\alpha \in \mathrm{Card}$ and $D \in S_{\alpha^+}^+$, set

$$R^D = \bigcup_{\delta \in (\alpha^+, \alpha^{++}(} R^{D \restriction \delta}.$$

Lemma 3. *Let $D \in S_{\alpha^+}{}^+$, then:*

(1) *R^D is (α, ∞) distributive in A_D,*

(2) *R^D satisfies the $\leqslant \alpha^+$ chain condition in A_D,*

if G is R^D generic over A_D, then there is a set $B \subset (\alpha, \alpha^+($ such that:

(3) *$A_D(G) = {}_{L_\alpha^{++}}(B)$ and for $\beta \in (\alpha, \alpha^{++}($ and ξ if $L_\beta(B \cap Z \cap \xi) \models \mathrm{ZF} + \xi = \alpha^+$, then $L_\beta(B \cap \xi) \not\models \mathrm{ZF}$,*

(4) *for $\delta \in (\alpha^+, \alpha^{++}($ G is $R^{D \restriction \delta}$ generic over $A_{D \restriction \delta}$.*

Proof. (1), (2), (3) are easy consequences of the Lemma 2; (4) comes from:

Lemma 4. *If $\Delta \subset R^{D \restriction \delta}$, $\Delta \in A_{D \restriction \delta}$ is predense in $R^{D \restriction \delta}$, then Δ is predense in R^D. (where predense means: $\{q \mid \exists\, p \in \Delta\; q \leqslant p\}$ is dense).*

Proof. This is exactly as the [9, Section 1, Lemma 2]; as it is very technical we omit it. This is why in the definition of R^s we introduce $P_{<\alpha^+}(\alpha^+)$ and the b_p have to be mutually generic. $\square$

Lemma 5. *There is a sequence $\langle c_\beta \mid \beta$ limit cardinal$\rangle$ such that:*

(1) *$c_\beta \subset \beta$ is closed in β.*

(2) *$\gamma \in c_\beta \to \gamma$ is a limit cardinal and $c_\gamma = \gamma \cap c_\beta$.*

(3) *c_β is of order type less than β.*

(4) $\operatorname{Sup} c_\beta < c_\beta \to \operatorname{cf} \beta = \omega$.

(5) *If $L_\mu \models \mathrm{ZF}^-$ and $\mu > \beta$, then $c_\beta \in L_\mu$ and is uniformly definable from β.*

Proof. See [9]. This use fine structure methods. Recall that in M every limit cardinal is singular. $\square$

Definition. Let α be a limit cardinal and $s \in S_\alpha$ and τ be a cardinal less than α $X_{s\tau}$ = the smallest $X \prec A_s$ such that $\tau \subset X$. $\pi_{s\tau} : A_{s\tau} \xrightarrow{\sim} X_{s\tau}$ where $A_{s\tau}$ is transitive. Note that each set in A_s is definable from parameters in α and so $A_s = \bigcup_{\tau<\alpha} X_{s\tau}$. $\tilde{\rho}_{s\tau}$ = the L-code of $A_{s\tau}$. (so $\tilde{\rho}_{s\tau} \in (\tau^+, \tau^{++}($).

Definition. Let α be a limit cardinal set:

$$\lambda_\alpha = \begin{cases} |c_\alpha - |c_\alpha|| & \text{if } \operatorname{Sup} c_\alpha = \alpha \,(|x| = \text{the order type of } x), \\ \omega & \text{otherwise.} \end{cases}$$

If $\operatorname{Sup} c_\alpha = \alpha$ set $\langle \gamma_i^\alpha \mid i < \lambda_\alpha \rangle$ = the monotone enumeration of $c_\alpha - |c_\alpha|$. If not set $\langle \gamma_i^\alpha \mid i < \lambda_\alpha \rangle$ = the L-least ω-sequence of successor cardinals converging to α such that $\gamma_0^\alpha > \operatorname{Sup} c_\alpha$.

Definition. Let α be a limit cardinal and $s \in S_\alpha$, set for $i < \lambda_\alpha$ $\rho_{si} = \langle\langle\tilde{\rho}_s \gamma_i^\alpha, |c_\alpha|\rangle, 2\rangle$. We now define sets of conditions P_τ^s (α, $\tau \in \mathrm{Card}$, $\tau \leqslant \alpha$, $s \in S_\alpha$) by induction as follows: P^s is the set of maps p: $(\tau, \alpha(\cap \mathrm{Card} \to M$ such that setting $p(\gamma) = \langle \dot{p}_\gamma, p_\gamma \rangle$ we have:

(1) if $\alpha = \gamma^+$, $\tau \leqslant \gamma$, then $p(\gamma) \in R^s$,

(2) $\forall\, \gamma \in \mathrm{Card} \cap (\tau, \alpha($ $p \restriction \gamma \in P_\tau^{p_\gamma}$,

(3) if limit α, $\tau < \alpha$, then $p \in A_s$ and if $|p|$ is the least $\xi \leqslant |s|$ such that $p \in A_{s \restriction \xi}$, then

$$\exists\, i < \lambda_\alpha\ \forall\, j < \lambda_\alpha\ (j \geqslant i \to \hat{p}(\rho_{s \restriction \xi, i}) = s(\xi)) \quad \text{where } \hat{p} = \bigcup_{\gamma \in (\tau, \alpha(\cap \mathrm{Card}} p_\gamma.$$

We set $p \leqslant q$ iff $p, q \in P_\tau^s$ and $\forall\, \gamma \in (\tau, \alpha(\cap \mathrm{Card}$ $\dot{p}_\gamma \supset \dot{q}_\gamma$ and $p_\gamma \supset q_\gamma$.

Note: For a motivation of the condition (3) see [9, Section 1, Theorem 4]. This is the way a subset of α^+ can be coded by a subset of α (the almost disjoint sets cannot be used for limit cardinals). We use the ρ_{si} to ensure that these coding are reasonably independent.

Lemma 6. *Let $s \in S_{\alpha^+}$; $\tau \leqslant \alpha$ and $\xi \in [\alpha^+, |s|)$; if $\Delta \in A_{s \restriction \xi}$ is predense in $P_\tau^{s \restriction \xi}$, then Δ is predense in P_τ^s.*

Proof. As in [9]; this a consequence of Lemma 4. □

Definition. Let $D \in S_\alpha^+$;

$$P_\tau^D = \bigcup_{\delta \in [\alpha, \alpha^+[} P_\tau^{D \restriction \delta}.$$

Lemma 7. *Let $D \in S_\alpha^+$, then P_τ^D satisfies the $\leqslant \alpha^+$ chain condition in A_D.*

Proof. An easy consequence of Lemma 3. □

Definition.

$$P_\tau = \bigcup_{\substack{\alpha \in \mathrm{On} \\ s \in S_\tau}} P_\tau^s = \bigcup_{\alpha \in \mathrm{On}} P_\tau^{\alpha^+}$$

(where $P_\tau^\gamma = P_\tau^s$ with $s \in S_\gamma$, $s = \emptyset$). $p \leqslant q$ iff $\exists s (p, q \in P_\tau^s$ and $p \leqslant q$ in $P_\tau^s)$. P_0 will be our ultimate class of conditions.

Definition. If $G \subset P_\tau^s$ is P_τ^s generic over A_s we set:

$$D_\gamma = \bigcup_{p \in G} \tilde{p}_\gamma$$

(where $\tilde{p}_\gamma = \{\xi \mid p_\gamma(\xi) = 1\}$).

$$D = \bigcup_{\gamma \in \mathrm{Card} \cap (\tau, \alpha)} D_\gamma \quad (\text{if } s \in S_{\alpha^+}).$$

Clearly

$$A_s(G) = L_{\mu_s}(D).$$

We may define the same notion for G a P_τ generic over M.

Lemma 8. *Let G be P^s_τ (resp. P_τ) generic over A_s (resp. M) and $\gamma \in [\tau, \alpha]$ (resp. $[\tau, \infty[$) then:*

(i) *$D \cap [\gamma, \alpha^+[$ (resp. $D \cap [\gamma, \infty[$) is P^s_γ (resp. P_γ) generic over A_s (resp. M),*
(ii) *$D \cap [\tau, \gamma[$ is $P^{D_\gamma}_\tau$ generic over A_{D_γ},*
(iii) *$D \cap [\tau, \gamma^+[$ is $P^{\gamma^+}_\tau$ generic over A_{γ^+}.*

Proof. (i) and (ii) are trivial; (iii) is an easy consequence of Lemma 6. □

We now have to prove that P_0 does the job.

(1) We show that the conditions may be extended arbitrarily: this is provided by Lemma 9.

(2) We show that P^s_τ is (τ, ∞) distributive in A_s (this is Lemma 10). This will prove that if N is a P_0 generic extension of M, then cardinals and cofinalities are preserved and there is an r such that for no α $L_\alpha(r) \models \mathrm{ZF}$. (This is Lemma 15.)

(3) We show that P_τ is (τ, ∞) distributive in M. This will prove that the P_0 generic extension of M satisfies ZF.

Lemma 9. *Let α, τ be cardinals, α limit, $s \in S_\alpha$, $p \in P^s_\tau$, $\langle \xi_\gamma \mid \gamma \in (\tau, \alpha(\cap \mathrm{Card\text{-}limit} \rangle \in A_s$ such that $\forall \gamma\ |p|_\gamma \leq \xi_\gamma < \gamma^+$; there is $q \in P^s_\tau$ $q \leq p$ such that:*

(1) $\forall\, \gamma \in \mathrm{Card}$-*limit* $\cap (\tau, \alpha(\ |q \restriction \gamma| = |q_\gamma|$,
(2) $|q| = |s|$,
(3) $\forall \gamma$ *limit* $|q_\gamma| \geq \xi_\gamma$.

From this lemma everything will be clear about the extension of the conditions.

Proof. We prove it by induction on α. So suppose this is true for the ordinals $\beta < \alpha$. We only give a sketch of the proof and say what has to be done: the precise details are in [9, where this is essentially Lemmas 12-7 and 12-8 of §2 and Lemma 4-1 of §1].

If α is a limit cardinal set:

$$U_\beta = \{\delta = \langle \langle \rho, |c_\beta| \rangle, 2 \rangle \mid \exists\, i < \lambda_\beta\ \delta \in (\gamma_i^{\beta^+}, \gamma_i^{\beta^{++}}(\}$$

(then if $s \in S_\beta$, $\rho_{si} \in U_\beta$).

Claim 1. (i) *If* $\beta' < \beta$, *then* $U'_\beta \cap U_\beta$ *is bounded in* β',
(ii) $\beta' \in c_\beta \rightarrow U_{\beta'} \cap U_\beta = \emptyset$.

This is easily shown by using Lemma 5.

Using the method developed in [9, §1, Lemma 4-4] we can find $q \leqslant p$ such that

$$|q| = |s| \;\&\; \forall\, \gamma \in (\tau, \alpha(\; \dot{q}_\gamma = \dot{p}_\gamma \;\&\; \hat{q} \restriction (\tau, \alpha(-U_\beta = \hat{p} \restriction (\tau, \alpha(-U_\beta.$$

Set $\langle \beta_i \mid i < \rho \rangle =$ the monotone enumeration of the limit points of $\langle \gamma_i^\alpha \mid i < \lambda_\alpha \rangle \cap (\tau, \alpha($; ($\rho$ may be any ordinal!).

Let $r(i) \in R^q \beta_i^+$ be such that

$$r(i) \leqslant q(\beta_i) \;\&\; |r_i| \geqslant \xi_{\beta_i} \;\&\; \langle \xi_\gamma \mid \gamma < \beta_i \rangle \in A_{r_i}.$$

Using the method in [9, §1, Lemma 4] and the properties stated by the Claim 1 we can find $t \leqslant q$ such that:

$$|t \restriction \beta_i| = |t_{\beta_i}| \quad (\text{for } i < \rho) \;\&\; t(\beta_i) \leqslant r(i).$$

(We extend q on the U_{β_i}, and use Z_4 to ensure this first property; since the U_{β_i} are independent and $\dot{r}_i$ affects only Z_1 it is an easy construction to do all that together.)

Now using the induction hypothesis we extend t 'along' the γ_i^α (what was difficult to ensure: the fact that for limit γ, $p \restriction \gamma \in A_{p_\gamma}$ has been provided just before):

If $\operatorname{cf} \alpha = \omega$, it is easy since there is no limit problem!

If $\operatorname{cf} \alpha > \omega$, then $\operatorname{Sup}_{i<\rho} \beta_i = \alpha$.

For $i < \rho$ set $v_i =$ the $A_{t_{\beta_{i+1}}}$ least $v \leqslant p \restriction (\beta_i, \beta_{i+1}($in $P_{\beta_i}^{t_{\beta_{i+1}}})$ such that

$$|v \restriction \gamma| = |v_\gamma| \text{ for limit } \gamma \in \mathrm{Card} \cap)\beta_i, \beta_{i+1}(\;\&\; |v_\gamma| \geqslant \xi_\gamma \;\&\; v_{\beta_i} = t_{\beta_i}.$$

It is then clear that $w = \bigcup_{i<\rho} v_i$ has the desired properties. $\square$

We now prove 'distributivity'.

Lemma 10. *Let* $s \in S_{\alpha^+}$, $\tau \leqslant \alpha$, *then* P_τ^s *is* (τ, ∞) *distributive in* A_s.

The proof of this lemma is by induction on α. It is rather fastidious and very technical. So we shall omit a big part of the details (as usual they can be found in [9]).

We shall only give the different steps of the proof and the ideas, stating the important lemmas and proving the only fact that is different from the proof in [9] (it is Lemma 15).

The reader has to keep in mind that we are doing something as an Easton forcing (to add a generic subset of $(\alpha, \alpha^+($ by conditions of size α for each α): but here the different stages are not independent. To give an idea of things that are proven we state first (this is [9, Lemma 1 of §3]).

Lemma 11. *Let $\tau \leqslant \gamma < \alpha$, $s \in S_{\alpha^+}$. Assume P_{γ^+} is (γ^+, ∞) distributive in A_s; let $\Delta_\nu \in A_s$ be strongly dense in P^s for $\nu < \gamma^+$. For $p \in P^s_{\gamma^+}$ set $\Delta^p_\nu = \{q \in P^p_\tau\gamma^+ \mid q \cup p \in \Delta_\nu\}$ and $\Delta = \{p \in P^s_{\gamma^+} \mid \forall\, \nu < \gamma^+\ \Delta^p_\nu$ is dense in $P^p_\tau\gamma^+\}$. Then Δ is dense in P^s.*

Lemma 12. *Suppose Lemma 10 holds for $\beta \leqslant \alpha$; let $s \in S_\alpha +$, $\tau \leqslant \alpha$, $D \subset (\tau, \alpha^+($ be P^s_τ generic over A_s; then cardinals and cofinalities are preserved in $A_s(D)$.*

Proof. By almost standard methods, using distributivity and chain condition. □

Lemma 13. *Suppose Lemma 10 holds for $\beta \leqslant \alpha$; let $s \in S_{\alpha^+}$, $\tau \leqslant \alpha$ and $D \subset (\tau, \alpha^+($ be P^s_τ generic over A_s then:*

(1) *s, $D_\gamma \in L\mu_s(D_\tau \cap Z)$ for $\gamma > \tau$,*

(2) *$\forall\, \beta \in (\tau, \mu_s(\ L_\beta(D_\tau \cap \beta) \nvDash \mathrm{ZF}$.*

Proof. (1) Clearly everything has been done so that it holds! (Since cardinals are preserved, from $D_\tau \cap Z$ we recover $D_\tau + \cdots$ etc. and for limit γ we recover D_γ by use of the property (3) in the definition of $P^s_\tau \cdots$ and at the end we recover s.

(2) If β is a cardinal this comes from the assumption on M. So suppose β is not a cardinal

- if $\beta > \alpha^+$: it comes from the fact that $s \in L_{\mu_s}(D_\tau \cap Z)$ and Lemma 2;
- if $\beta \in)\tau, \tau^+($: it comes directly from the definition of S_τ;
- if $\beta \in)\tau^+, \alpha^+($: suppose $L_\beta(D_\tau) \vDash \mathrm{ZF}$ and $\bar{\bar{\beta}} = \gamma$; let δ be such that $L_\beta(D_\tau) \vDash \delta = \gamma^+$. It is enough to show that $D \cap (\gamma, \delta(\in L_\beta(D_\tau)$ (since by the definition of S_γ $L_\beta(D \cap (\gamma, \delta() \nvDash \mathrm{ZF}$ because $D \cap (\gamma, \delta(\in S_\gamma)$.

But this is an easy consequence of the proof of the part (1) of this lemma. □

We now give the proof of Lemma 10.

If α is a successor cardinal this is an easy consequence of Lemma 12 and the induction.

If α is limit (note that we have stated the lemma before the proof of Lemma 10 because it will be used in the proof!): Let $p \in P^s_\tau$ and $(\Delta_i)_{i<\tau} \in A_s$ be a family of strongly dense subsets of P^s_τ we shall define a family $(p_i)_{i \leqslant \tau}$ and $Y^\gamma_i < A_s$ ($\gamma \in \mathrm{Card} \cap (\tau, \alpha)$, $(i \leqslant \tau)$) such that the following hold:

(i) $\gamma \subset Y^\gamma_i$ & $Y^\gamma_\lambda = \bigcup_{i<\lambda} Y^\gamma_i$ if limit λ,

(ii) $p_{i+1} \in \Delta_i$ & $p_{i+1} \leqslant p_i$,

(iii) if limit λ, $\gamma \in (\tau, \alpha)$ then $p_\lambda = \bigcup_{i<\lambda} p_i$ and if we set:

$\sigma : b \xrightarrow{\sim} Y^\gamma_\lambda$, b transitive; $b = L_{\mu'}(s')$; $s' = \sigma^{-1}(s)$; $p'_i = \sigma^{-1}(p_i)$; $\alpha' = \sigma^{-1}(\alpha)$. For δ such that $b \vDash \delta \in \mathrm{card} \cap (\gamma, \alpha')$ set

$$D'_\delta = \bigcup_{i<\lambda} (p'_i)_\delta \quad \text{and} \quad D' = \bigcup D'_\delta.$$

If P' is defined in b as P was defined in A_s, then:

(1) D' is $p'^{s'}_\gamma$ generic over b,

(2) $D'_\gamma = (p_\lambda)_\gamma$ (here with no prime!).
(This fact can be ensured by meeting enough dense subsets, not only in P^s_τ, but also of things like those in Lemma 11; here is the technical and fastidious—but completely *not* trivial!—fact.)

As usual the only thing that has to be proved is that p_λ is a condition for limit λ set $p = p_\lambda$; the only non trivial fact is:

Lemma 14. (1) *For* $\gamma \in \mathrm{Card} \cap (\tau, \alpha)$ $p_\gamma \in S_\gamma$ *and more precisely*: *if* $\xi = |p_\gamma|$ & $\beta > \xi$ & $L_\beta(p_\gamma) \models \mathrm{ZF}$, *then* $L_\beta(p_\gamma \cap Z) \models \bar{\bar{\xi}} = \gamma$.
(2) $p \restriction \gamma \in A_{p_\gamma}$.

Proof. (1) Since D' is $P^{\gamma'}_\gamma$ generic over b, using Lemmas 12 and 13 (1) (relativized to b) it is easily shown that: $b, (p'_i)_{i<\lambda} \in L_{\mu'+\omega}(p_\gamma \cap Z)$. Suppose then $\beta > \xi$ and $L_\beta(p_\gamma) \models \mathrm{ZF}$
– if $\beta > \mu'$: then $\beta > \mu' + \omega$ (since $L_\beta \models \mathrm{ZF}$) but then $b, (p'_i)_{i<\lambda} \in L_\beta(p_\gamma \cap Z)$ and so $L_\beta(p_\gamma \cap Z) \models \bar{\bar{\xi}} = \gamma$;
– if $\beta = \mu'$: it is impossible since $L'_\mu(p_\gamma)$ satisfies: there is a greatest cardinal;
– if $\beta < \mu'$: Lemma 13 (relativized to b) shows that we cannot have $L_\beta(p_\gamma) \models \mathrm{ZF}$.
(2) It is proved from the fact that $(p'_i)_{i<\lambda} \in L_{\mu'+\omega}(p_\gamma)$ and showing that $\mu' < \mu_{p_\gamma}$. □

Note 1. Here we use the fact that it is all ZF that we have to put in the definition of S_α and not only a finite part: because of the reflection principle, in Lemma 13(2) we cannot say more than $L_\beta(D_\tau \cap Z) \not\models \mathrm{ZF}$. For β *not* a cardinal of course we could assume it is always the same finite part of ZF that is used, but not for a cardinal! Since we use this lemma to prove the claim before (relativized to b, and so where ordinals that are not really cardinals seem to be cardinals) we cannot suppose (in the definition of S_α) that it is a finite part of ZF that is used!

Note 2. In fact the precedent proof is used in [9] for the case α inaccessible or $\mathrm{cf}\,\alpha = \omega$; the case $\mathrm{cf}\,\alpha > \omega$ is just a bit different (for technical reasons) but the idea is the same and the proof too.

Lemma 15. *Let* τ *be a cardinal, D be* P_τ *generic over M and* $N = M(D)$ *then*:
(i) $N \models V = L(D_\tau)$,
(ii) $\forall \alpha \geq \tau\, L_\alpha(D_\tau \cap \alpha) \not\models \mathrm{ZF}$,
(iii) *cardinals and cofinalities are preserved and* $H_\alpha = L_\alpha(D_\tau)$.

Proof. It is enough to prove (iii); (i) and (ii) will follow by the same proof as in Lemma 13. So let $\alpha \leq \beta$ be cardinals in M; then (by Lemma 8) $D \cap \beta^+$ is $P^{\beta^+}_\tau$ generic over A_{β^+} but (by Lemma 12) in $A_{\beta^+}(D \cap \beta^+)$: α is a cardinal; $\mathrm{cf}(\alpha) = \mathrm{cf}_M(\alpha)$; $H_\alpha = L_\alpha(D)$ but this holds for arbitrarily large β and

$$N \subset \bigcup_{\beta \in \mathrm{Card}_M} A_{\beta^+}(D \cap \beta^+)$$

and so we are done. □

Lemma 16. *P_τ is (τ, ∞) distributive in M.*

Proof. This is virtually the same as in Lemma 10; we repeat the proof of Lemma 10 using a Σ_n restriction of M (n large enough) instead of an elementary restriction of A_s, and Lemma 15 instead of lemmas 12 and 13. $\square$

Now it remains to prove that if N is a P_τ generic extension of M, then N satisfies ZF: Lemma 15(iii) shows that the Power set axiom holds in N. The Replacement axiom comes from Lemmas 7, 8 and 16 by entirely standard Easton methods.

This achieves the proof of Theorem 1.

We now give some details on the proof of the Theorems 3 and 4.

3. The complements

Proof of Theorem 4. It is clear that in the proof of Theorem 1 we can start everything by $\aleph_2$; so we can assume that we have a generic extension M_0 of M that satisfies:

(1) $\mathrm{ZF} + \exists\, A \subset \aleph_2 V = L(A)$.

(2) $\forall x (x \in \aleph_1 \to x \in L)$.

(3) $\forall\, \beta > \aleph_2\, L_\beta(A) \nvDash \mathrm{ZF}$.

(4) $\forall\, \xi < \aleph_2\, \forall \beta\, L_\beta(A \cap \xi) \vDash \mathrm{ZF} \to L_\beta(A \cap Z_0 \cap \xi) \vDash \bar{\bar{\xi}} \leqslant \aleph_1)$.

To ensure that the final real is Π^1_2 we use the sequence $(T_n)_{n \in \omega}$ of Suslin trees constructed in [10] (also see [2] or [7]).

Starting with M_0 we have to:

(i) code A by a subset B of $\aleph_1$,

(ii) use the forcing R to ensure there is no α in $\aleph_1$ such that

$$L_\alpha(B \cap \alpha) \vDash \mathrm{ZF},$$

(iii) code that by a real which gives branches in the trees T_n and so is Π^1_2.

The forcing in (i) and (ii) have to preserve the suslinity of the trees; it is true for the first one (we proved this in [2] using an inversion of forcing and the next lemma).

Lemma. *Let M be a model of ZF, T a Suslin tree in M and Q a notion of forcing which is $\aleph_1$ closed (i.e. every countable descending chain of condition has a minorant); if N is a Q generic extension of M, then T still is Suslin in N.*

(*Note*: In fact it is only needed that the following — that seems weaker — is true: Let $(\Delta_i)_{i<\aleph_1}$ be a sequence of strongly dense subset of Q, there is a sequence $(p_i)_{i<\aleph_1}$ of conditions such that: $\forall\, i, j < \aleph_1\ (j > i \to p_j \in \Delta_i)$.

But it occurs that the forcing R is only (ω, ∞) distributive — and does not satisfy the property before. So we have to be more careful. Starting with M_0 we first code

A by a subset B of $\aleph_1$; it is proved that in $M_0(B)$ the forcing Q (with the Suslin trees) has the c.c.c. (countable chain condition) (for the proof see [2]).

Then we code B by a real a, using the almost disjoint sets forcing 'married' with forcing Q. Let M_1 be the generic extension. We know that there is a Π^1_2 formula φ_0 such that M_1 satisfies:

(i) $V = L(a) + \varphi_0(a)$;

(ii) $\forall \alpha > \aleph_1 \quad L_\alpha(a) \nvDash \mathrm{ZF}$;

(iii) $\forall x(\varphi_0(x) \to x = a)$.

Moreover if M_2 is an extension of M_1 that preserves $\aleph_1$, then (iii) remains true in M_2.

Now define in $M_1 = L(a)$ Suslin trees $(\tilde{T}_n)_{n\in\omega}$ as T_n were defined in L. Let $\tilde{Q}$ be the associated forcing; we know that $\tilde{Q}$ has the c.c.c. in M_1. Before giving the next forcing we must recall some notations of the construction in [10] of the trees $\tilde{T}_n$.

If $(\tilde{T}_n \restriction \alpha)_{n<\omega}$ is constructed (limit α), to define $\tilde{T}_{n+1}$ at the level α we use a forcing over some $L_{\eta_{n+1,\alpha}}(a)$ where $\eta_{n+1,\alpha}$ is the first ordinal η such that:

$$T_n \restriction \alpha + 1,\ T_{n+1} \restriction \alpha \in L_\eta(a)$$

$$L_\eta(a) \vDash \mathrm{ZF}^- + \bar{\bar{\alpha}} = \omega.$$

Set $\eta_\alpha = \eta_{0,\alpha}$, we now define $\tilde{R}$ as follows: $p \in \tilde{R}$ iff $p : |p| \to 2$ such that $|p| < \aleph_1$ and for $\xi \leqslant |p|$ and β

(i) ξ limit $\to p \restriction \xi \in L_{\eta_\xi}(a)$,

(ii) $L_\beta(a, p \restriction \xi) \vDash \mathrm{ZF} \to L_\beta(a) \vDash \bar{\bar{\xi}} = \omega$.

Claim. (i) $\forall p \in \tilde{R}\ \forall \xi \in [|p|, \aleph_1[\ (\exists\, q \leqslant p\ |q| \geqslant \xi)$.

(ii) *R is (ω_1, ∞) distributive in M_1.*

Proof. (i) is easy; the proof of (ii) is exactly as in Lemma 2(1): we simply use the fact that for limit λ: $\eta_{\alpha_\lambda} > \delta_\lambda$. $\square$

Claim. *Let D be $\tilde{R}$ generic over M_1, then Q has the c.c.c. in $M_1(D)$.*

Proof. We follow exactly the proof in $L(a)$ and use the fact that $D \cap \xi \in L_{\eta_\xi}(a)$.

Now let M_2 be a $\tilde{R}$ generic extension of M_1, and code $a \cup D$ by a real r using the almost disjoint sets forcing 'married' with the forcing $\tilde{Q}$ (we may assume that: $n \in a \leftrightarrow S(n) \cap r$ is finite). We say that r does the job. It is clear that: $\forall \alpha\ L_\alpha(r) \nvDash \mathrm{ZF}$. We have to prove that r is Π^1_2 in $M_3 = M_2(r) = L(r)$: we use the same idea as in [3]. (We cannot conclude immediately since to be Π^1_2 in a Π^1_2 is not the same as to be Π^1_2!)

As in [10] the formula φ_0 comes from a Π_1^{ZF} formula ψ_0 such that:

$$\varphi_0(a) \leftrightarrow H \vDash \psi_0(a) \quad (H = \text{the hereditary countable sets})$$

ψ_0 is:

$$\forall\alpha\,\forall T\,(T = \Pi_n\, T_n \mid \alpha \rightarrow \theta_0(\alpha, t, x)).$$

Where θ_0 is Δ_1 and says: $\exists\, p \in T\, p = (p_n)_n$ & $p_n \geqslant_{T_n}$ the nth element of x & something to ensure this will give branches. The formula $T = \Pi_n\, T_n \mid \alpha$ also is Σ_1. More generally if y is a real and $\bar{T}_n$ are calculated in $L(y)$ we have Δ_1 and Σ_1 formulas $\theta_1(\alpha, T, x, y)$ and $T(y) = \Pi_n\, T_n \mid \alpha(y)$.

Let ψ_1 be the following formula:

$$\forall\alpha\,\forall T\,\forall y\,\{(n \in y \leftrightarrow S(n) \cap x \text{ is finite}) \;\&\; T(y) = \Pi_n\, T_n \mid \alpha(y) \rightarrow$$
$$\rightarrow\; \theta_1(\alpha, t, x, y)\}$$
$$\&\; \forall y\{n \in y \leftrightarrow S(n) \cap x \text{ is finite}) \rightarrow \psi_0(y)\}.$$

It is clear that ψ_1 is Π_1^{ZF} and that: $H \vDash \psi_1(x) \leftrightarrow x = r$. This achieves the proof. $\square$

Remark. We prove this theorem starting with a model of ZF+ $V = L$. In fact we may start with much more general models (but not all!).

(1) It can be shown that the result is true for a model M which is a generic extension (by a *set* of conditions) of a model of $V = L$. The idea is: let K be the cardinal of the set of conditions; construct in L K^{++} Suslin trees T_n as in [10], using instead of Q an elementary extension of the rationals which is of cardinal K^+ and saturated (the functions $f_n : T_{n+1} \rightarrow T_n$ cannot be defined but since they can be recovered from the real by a coding it does not matter); then collapse K^+ on ω; in the extension K^{++} becomes $\aleph_1$ and the trees are $\aleph_1$ Suslin and definable in L.

(2) The result can also be true for generic extension of L by a class of conditions we must be able to prove that some Suslin trees defined in L remain Suslin in the extension. For example if we start from L and add, by an Easton forcing, a set $A_\alpha \subset (\alpha, \alpha^+($ for every cardinal α by conditions of size α, the result will be true. But if M has the property: there is no inaccessible cardinal & $\forall\, \alpha \in \mathrm{On}\, \aleph_{\alpha+2}$ is inaccessible in L (this can be done by using the method developed in Theorems 6 and 7), then I do not know if we can have the conclusion of Theorem 4.

(3) If we start from a model of ZF+ $V = L(0^{\#})$ (or ZF+ $V = L(0^{\#\#})$ or else $V = L^{\#}$) we can also have the conclusion by using the methods developed in [3].

(4) The corollary of Theorem 4 gives a big improvement to the theorem of Barwise (see [1]) where the extension has not the same ordinals.

Proof of Theorem 3. For $i \leqslant \lambda$ set $\beta_i = \mathrm{Sup}_{j<i}\, \beta_j$ (set $\beta_0 = \alpha$) (so if $i = j + 1$ $\beta_i = \alpha_j$).

For $i < \lambda$ let γ_i be the least cardinal (in the sense of L_{α_i}) greater than β_i

$$\begin{array}{ccccccccc} & & & \alpha_0 & & \alpha_1 & & & \alpha_\omega \\ \hline 0 & \beta_0 & \gamma_0 & \beta_1 & \gamma_1 & \beta_2 \;\cdots\cdots & \beta_\omega & \gamma_\omega & \beta_{\omega+1} \end{array}$$

For $i \leq \lambda$ set

$$Q_i = \left\{ p \mid \operatorname{dom} p \in \omega \;\&\; p : \operatorname{dom} p \to \bigcup_{j<i} [\gamma_j, \alpha_j[\cup \{\infty\} \right\}$$

$p \leq q$ iff $p \supset q$.

For $i \leq j \leq \lambda$ and $p \in Q_j$ define p_i by: dom p_i = dom p and

$$p_i(n) = \begin{cases} p(n) & \text{if } p(n) < \beta_i, \\ \infty & \text{if not.} \end{cases}$$

Let G be Q_λ generic over $L_{\beta_\lambda+1}$; set $A = \{(n, \xi) \mid n \in \omega \;\&\; \xi \in \beta_\lambda \;\&\; G(n) = \xi\}$.

Claim 1. *For $i < \lambda$ $L_{\alpha_i}(A \cap \beta_i) \models \mathrm{ZF} + \gamma_i = \aleph_1$.*

$G_i = \{p_i \mid p \in G\}$ is clearly Q_i generic over L_{α_i}.

For $i < \lambda$ let P_i be the forcing (calculated in $L_{\alpha_i}(A \cap \beta_i)$) developed in the proof of Theorem 1 to add a subset B_i of $[\beta_i, \gamma_i[$ such that:

$$(*) \qquad \begin{array}{l} \forall \delta \in [\gamma_i, \alpha_i[\; L_\delta(A \cap \beta_i, B_i) \not\models \mathrm{ZF}, \\ \forall \xi \in [\beta_i, \gamma_i[\; \forall \delta \; (L_\delta(A \cap \beta_i, B_i \cap \xi) \models \mathrm{ZF} \Rightarrow L_\delta(A \cap \beta_i) \models \bar{\bar{\xi}} = \omega). \end{array}$$

For $i \leq \lambda$ set

$$P_i^+ = \prod_{j<i} P_j = \{(p_j)_{j<i} \mid \{j \mid P_j \neq 1 p_j\} \text{ is finite}\}.$$

Let $(B_i)_{i<\lambda}$ be P_λ generic over $L_{\beta_\lambda+1}(A)$; set

$$C = A \cup \bigcup_{i<\lambda} B_i.$$

Claim 2. *For $i < \lambda$ $L_{\alpha_i}(C \cap \gamma_i) \models \mathrm{ZF}$.*

By the hypothesis $P_i^+ \in L_{\alpha_i}(A \cap \beta_i)$; $(B_j)_{j<i}$ is P_i^+ generic over $L_{\alpha_i}(A \cap \beta_i)$, so $L_{\alpha_i}(C \cap \beta_i) \models \mathrm{ZF}$; but $L_{\alpha_i}(C \cap \gamma_i) = L_{\alpha_i}(C \cap \beta_i)(B_i)$ and the result follows from the proof of Theorem 1.

Claim 3. *For $\xi < \beta_\lambda$ if i is the least such that $\xi < \gamma_i$, then $L_{\alpha_i}(C \cap \xi) \models \bar{\bar{\xi}} = \omega$.*

If $\xi \in [\beta_i, \gamma_i[$ this is because $L_{\alpha_i}(A \cap \beta_i) \models \gamma_i = \aleph_1$.

If not, then $i = j + 1$ and $\xi \in [\gamma_j, \alpha_j[$: then $L_{\alpha_i}(C \cap \gamma_j) \models \bar{\bar{\alpha}}_j = \omega$ (if not an application of the Lowenheim–Skolem heorem to $L_{\alpha_i}(C \cap \gamma_j)$) gives a contradiction with the property $(*)$ of B_i).

Now by Claim 3 we can code C by a real r (by use of almost disjoint sets) so that r is generic over $L_{\alpha_i}(C \cap \gamma_i)$ for $i < \lambda$. (The proof is as in [9, §1, Lemma 2].) Now r clearly does the job.

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